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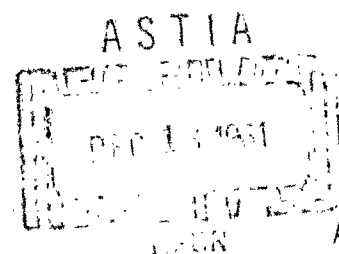
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A GEOMETRIC INTERPRETATION OF

PONTRJAGIN'S MAXIMUM PRINCIPLE

By
Emilio Roxin

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1. Introduction

In connection with problems of control mechanisms, there have been studied systems of differential equations in which some more or less arbitrary "control functions" appear. An important question is how to choose those control functions in order to minimize (or maximize) a certain functional of the solution. Physically speaking, this may mean to operate the system in the "best" possible way, minimizing a "cost" function or simply the time of operation [1].

Pontrjagin and his collaborators [2] have pointed out a general principle (the "maximum principle"), which applies to this problem.

The problem of what can be accomplished by adequate election of the control function, disregarding the optimization problem, has been studied independently by Kalman [3] and the author [4], [5].

In this paper we want to look at these problems from a geometrical point of view; our aim is to show some interesting relations which remain hidden in a more analytical treatment, like that of Pontrjagin, Boltyanskii and Gamkrelidze.

We shall study the system

(1.1) $$x^1 = f^1(t, x^1, \ldots, x^n, u^1, \ldots, u^m); \quad i = 1, 2, \ldots, n,$$

where, as usual $\dot{x} = dx/dt$. Let us write

(1.2) $$x^o = t, \qquad f^o = 1$$

and introduce the following vectors

$$\hat{x} = (x^1, \ldots, x^n)$$

$$x = (x^o, x^1, \ldots, x^n) = (x^o, \hat{x})$$

(1.3) $$\hat{f} = (f^1, \ldots, f^n)$$

$$f = (f^o, f^1, \ldots, f^n) = (f^o, \hat{f})$$

$$u = (u^1, \ldots, u^m).$$

These vectors are considered in euclidean vector spaces, with the usual norm: $\|x\|^2 = \Sigma(x^i)^2$. Equation (1.1) becomes, then,

(1.4) $$\dot{\hat{x}} = \hat{f}(t, \hat{x}, u)$$

or

(1.5) $$x = f(x, u).$$

We make the following assumptions:

i) $f(x, u)$ is defined in $X \times U$, where X is the x-space (the half-space corresponding to $x^o = t \geq 0$ could also be chosen) and U is a compact set of the u-space.

ii) $f(x, u)$ is continuous in (x, u).

iii) The Lipschitz condition holds:

$$\|f(x_2, u) - f(x_1, u)\| < K \cdot \|x_2 - x_1\|.$$

iv) $\|f(x, u)\| = O(\|x\|)$ for $\|x\| \to \infty$. This assumption is made in order to avoid a finite escape-time for the solutions.

We shall consider the parameter u as a function of t. $u(t)$ will be called an "admissible control function" if

v) $u(t)$ is measurable,

vi) for each t, $u(t) \in U$.

vii) "Convexity condition": $f(x, U) = \{f(x, u) | u \in U\}$ is a convex set for each x.

We note that being U compact and $f(x, u)$ continuous, the set $f(x, U)$ is compact too.

According to well known theorems (see [6], [7]), these conditions guarantee the existence and uniqueness of the solution of (1.5) for any admissible control function $u(t)$. The importance of the convexity condition (vii) will appear in the theorem stating that the set of reachable points is a closed set.

2. Reachable set.

Given the equation (1.5), we shall say that the point x_1 is reachable from x_o, if there exists an admissible control function $u(t)$ defined in the time interval $(t_o, t_1) = (x_o{}^o, x_1{}^o)$ where $t_1 \geq t_o$ such that equation (1.5), where $u = u(t)$, together with the initial condition

$$x(t_o) = x_o$$

has the solution $x(t)$ satisfying

$$x(t_1) = x_1.$$

The set of all points x which are reachable from x_o, will be called "reachable set from x_o", and denoted by R_{x_o}. When there is no possibility of misunderstanding, the subindex x_o will be dropped.

In many simple cases it is easy to see which points are reachable and which are not; the linear case has been treated in detail in [1], [8]. Here we are concerned with rather general properties of the reachable set.

Theorem 2.1: If the equation (1.5) fulfills the conditions (i) to (vii), then for any given initial x_o, the reachable set R_{x_o} is a closed set.

The proof of this theorem is given in [9]. A similar theorem under less general conditions is proved in [10]. The importance of this theorem lies in the fact that it may be used as an existence theorem for solutions of optimality problems like those which are the concern of Pontrjagin's Maximum Principle.

3. Some properties of the boundary of the reachable set.

It is important to study the properties of the boundary ∂R in the space $X = \{(x^0, x^1, \ldots, x^n)\}$, because there we might expect to find the solution of problems of the variational type. Let us take three examples.

a) Minimum time for reaching a given set $\hat{G}$ of the space $\hat{X} = \{(x^1, \ldots, x^n)\}$.

Let us start at $x_o = (x_o^0, x_o^1, \ldots, x_o^n)$, where $x_o^0 = t_o$; we want to reach any point $x_1 = (x_1^0, x_1^1, \ldots, x_1^n)$, such that $\hat{x}_1 = (x_1^1, \ldots, x_1^n) \in \hat{G}$ and $t_1 = x_1^0$ has the least possible value. Let G be the cylindrical set $I^+ \times \hat{G}$, where I^+ is that part of the x^0 axis, for which $t \geq t_o$. Then, what we are looking for is a point $x_1 \in R \cap G$ which minimizes $x_1^0 = t_1$. This point will belong to the boundary of R, because in every neighborhood of it there are points of G with smaller x^0 coordinates, which cannot belong to R because they do not belong to $G \cap R$.

b) If, in the preceding example, instead of minimizing the time $t_1 = x_1^0$, we want to minimize some function of the coordinates

$\psi(x_1^o, x_1^1, \ldots, x_1^n)$ then some additional assumptions have to be made. If on the set G, the function $\psi(x)$ has some minimum, attained at the set $G^* \subseteq G$, and if $G^* \cap R \neq \phi$ (some part of G^* is reachable), then any point of $G^* \cap R$ gives us a solution of our problem. These points do not need to belong to the boundary of R, and this case is called "degenerate" by Rozonoer [11], because Pontrjagin's principle does not apply to it. In fact (as we shall see), this principle applies if $x_1 \in G^* \cap R$ belongs to the boundary ∂R.

   c) If we want to minimize some functional of the type

(3.1) $$\int_{t_o}^{t_1} \varphi(x(t))dt,$$

the classical procedure consists in adding one coordinate to the X-space, putting

$$\dot{x}^{n+1} = \varphi(x); \qquad x^{n+1}(0) = 0,$$

and minimizing $\psi(x_1) = x_1^{n+1}$.

   In the example (b) it is also possible to write $x^{n+1} = \psi(x^o, x^1, \ldots, x^n)$ and minimize x^{n+1}, but in that case R is contained in the hypersurface $x^{n+1} = \psi(x^o, \ldots, x^n)$, so that R coincides with its boundary and those properties which depend on the fact that the solution point belongs to ∂R, become trivial.

   In this case, the value of the integral (3.1) depend in general on the trajectory $x(t)$, so that the set R does not need to be contained in some hypersurface. If we want to reach the set G we have, therefore, to look for the points on $G \cap R$ with minimum x^{n+1}. These are, of course, on the boundary of R because the set G is cylindrical with respect to the coordinate x^{n+1}.

   We now pass to establish a basic property of the trajectories belonging to the boundary of R.

Theorem 3.1: If for equation (1.5), R is the reachable set from the point ξ_0, if the point $\xi_1 \in \partial R$ and if $x_1(t)$ $(t_0 \leq t \leq t_1)$ is a trajectory going from $x_1(t_0) = \xi_0$ to $x_1(t_1) = \xi_1$, which corresponds to an admissible control $u_1(t)$ $(t_0 \leq t \leq t_1)$, then the whole trajectory $x_1(t)$ $(t_0 \leq t \leq t_1)$ belongs to the boundary of R.

The fact that $\xi_1 \in R$ guarantees the existence of the functions $u_1(t)$ and $x_1(t)$ mentioned. Let us assume that for $t = t_2$, $(t_0 \leq t_2 \leq t_1)$, the point $\xi_2 = x_1(t_2)$ is interior to R. There exists, therefore, a neighborhood A_2 of ξ_2, contained in R. Consider the equation

$$(3.2) \qquad \dot{\xi} = f(\xi, u_1(t))$$

in the interval (t_2, t_1), where now the control function $u_1(t)$ is fixed. For the initial condition $\xi(t_2) = \xi_2$ we obtain $\xi(t_1) = \xi_1$. Thus there is a certain neighborhood A_1 of ξ_1 such that $\xi(t_1) \in A_1$ implies $\xi(t_2) \in A_2$. We shall now prove that $A_1 \subseteq R$, which contradicts the fact that $\xi_1 \in \partial R$.

Let ξ_1' be any point of A_1. Equation (3.2) together with $\xi(t_1) = \xi_1'$ defines a trajectory such that $\xi(t_2) = \xi_2' \in A_2$. Since $\xi_2' \in R$, there exists an admissible control function $u_2(t)$ defined in the interval (t_0, t_2') (where $t_2' = \xi_2'^{0}$), such that

$$(3.3) \qquad \dot{x}_2(t) = f(x_2(t), u_2(t)); \quad x_2(t_0) = \xi_0$$

gives $x_2(t_2') = \xi_2'$.

We now extend this trajectory, coupling it with the path given by (3.2), in the following manner:

$$(3.4) \qquad \left. \begin{array}{l} u_2(t) = u_1(t_2 + t - t_2'); \\[2ex] \dot{x}_2(t) = f(x_2(t), u_2(t)); \quad x_2(t_2') = \xi_2' \end{array} \right\} \quad t_2' \leq t \leq t_1'$$

As this trajectory coincides with the former $\xi(t)$ (,,e have shifted the t but not the x-values), we obtain $x_2(t_1) = \xi_1'$ (see Figure 1). Hence, $\xi_1' \in R$ as we wanted to prove.

Figure 1

The variational and the adjoint equation.

It is well known that, for a fixed control function $u*(t)$ and the corresponding solution $x*(t)$ of equation (1.5), which we suppose given in the time interval (t_1, t_2), the neighboring solutions of (1.5) can be represented by the variational equation

$$(3.5) \qquad \dot{y} = [\frac{\partial f}{\partial x}]_{\substack{u = u*(t) \\ x = x*(t)}} \cdot y \; .$$

Here

$$(3.6) \qquad y(t) = x(t) - x*(t) + o(\|x(t_1) - x*(t_1)\|) \quad (')$$

is the first variation of $x*(t)$ and $\frac{\partial f}{\partial x}$ is the jacobian matrix.

Since the equation (3.5) is linear, its solution is of the form

$$(3.7) \qquad y(t) = A(t, t_0) \cdot y(t_0),$$

where $A(t, t_0)$ is a linear operator (i.e., a matrix).

$(')$ $y = o(x)$ means that $\lim \frac{y}{x} = 0$.

The y is a contravariant (column) vector, of the tangent space, and if $p = p(t)$ is a covariant (row) vector such that

$$(3.8) \qquad p(t) \cdot y(t) = \text{const.,}$$

it is known that $p(t)$ satisfies the adjoint equation

$$(3.9) \qquad \dot{p} = -p \cdot [\tfrac{\partial f}{\partial x}]_{\substack{u = u^*(t) \\ x = x^*(t)}} \, ,$$

the solution of which is

$$(3.10) \qquad p(t) = p(t_0) \cdot A(t_0, t).$$

This vector p is very useful for defining hyperplanes in the tangent space, as we shall see.

Theorem 3.2: If, for equation (1.5), R is the reachable set from a certain point ξ_0, if $x^*(t)$, $(t_1 \leq t \leq t_2)$ is an admissible trajectory of (1.5) belonging to the boundary of R and if at the points of $x^*(t)$ the boundary of R is a smooth surface with a tangent plane given by $p(t) \cdot (x - x^*(t)) = 0$, then $p(t)$ satisfies equation (3.9).

Proof: Let $x(t)$ $(t_1 \leq t \leq t_2)$ be a neighboring trajectory of $x^*(t)$, which corresponds to the same control function $u^*(t)$. If $x(t_1) \in R$ then evidently $x(t) \in R$ for any $t \in (t_1, t_2)$. The fact that $p \cdot (x - x^*)$ is the tangent plane means that

$$p \cdot (x - x^*) + o(\|x - x^*\|) \geq 0 \qquad \text{for } x \in R$$

and

$$p \cdot (x - x^*) + o(\|x - x^*\|) \leq 0 \qquad \text{for } x \notin R$$

where we have supposed, for example, that the set R is on that side of the tangent plane where the scalar product is positive. As $o(\|x - x^*\|)$ refers to $x \to x^*$, we are interested in a first order approximation and may substitute $x - x^*$ by y as given in (3.6). So, if $x(t_1) \in R$ and $y(t_1) = x(t_1) - x^*(t_1)$, we have

$$(3.11) \qquad p(t) \cdot y(t) + o(\|y(t_1)\|) \geq 0 \quad (t_1 \leq t \leq t_2)$$

and this determines completely the position of the tangent plane given by $p(t)$, since the mapping $y(t_1) \to y(t)$ is linear and transforms a half space into a half space. Besides, the vector $p^*(t)$ defined by equation (3.9) and the initial value $p^*(t_1) = p(t_1)$ also satisfies the relation (3.11) because $p^*(t) \cdot y(t) = \text{const.}$, so that identically $p^*(t) = p(t)$.

4. Tangent cone to a point set.

In order to simplify the statement of the properties of a reachable set, we introduce the concept of tangent cone to a given point set and start proving some elementary relations.

Definition 4.1: Given a point set A of the space X and a point $x_0 \in X$, we shall call tangent cone of the set A at the point x_0, the set of vectors y such that there exist a sequence of positive numbers $c_i (i = 1, 2, 3, \ldots)$ and a sequence of points $x_i \in A$ such that

$$\lim_{i \to \infty} x_i = x_0,$$

$$(4.1)$$

$$\lim_{i \to \infty} c_i (x_i - x_0) = y.$$

We shall denote the tangent cone of the set A at the point x_0 by $T_{x_0}[A]$, or simply by $T[A]$ when there is no possibility of misunderstanding.

According to the definition, if x_0 is an exterior point of the set A, the cone $T_{x_0}[A]$ will be empty. If x_0 is an interior point of A, it will be the whole tangent space (all vectors y). If the set A is a differentiable r-dimensional hypersurface and $x_0 \in A$, then the cone $T_{x_0}[A]$ is the r-dimensional tangent linear space.

Lemma 4.1: If $A = A_1 \cup A_2$, then

$$T[A] = T[A_1] \cup T[A_2].$$

Proof: Obviously, if $A_1 \subseteq A$, also $T[A_1] \subseteq T[A]$. Therefore

$$T[A] \supseteq T[A_1] \cup T[A_2].$$

For proving the opposite relation, it is sufficient to see that if $y \in T[A_1 \cup A_2]$, there exist c_i, x_i such that

$$y = \lim c_i(x_i - x_0)$$

with $x_i \in A_1 \cup A_2$, then some infinite subsequence x_{i_j} belongs to one or both sets A_1 or A_2.

Lemma 4.2: The tangent cone of any set is closed.

Proof: We have to show that if

(4.2) $$y = \lim_{i \to \infty} y_i$$

and

(4.3) $$y_i = \lim_{j \to \infty} c_{ij}(x_{ij} - x_0) \qquad i = 1, 2, 3, \ldots$$

where

(4.4) $$c_{ij} > 0; \quad x_{ij} \in A; \quad \lim_{j \to \infty} x_{ij} = x_0,$$

then there exist some sequences

$$\gamma_\alpha > 0, \quad x_\alpha \in A, \quad \alpha = 1,2,3, \ldots$$

such that

(4.5) $$\lim_{\alpha \to \infty} x_\alpha = x_0; \quad \lim_{\alpha \to \infty} \gamma_\alpha(x_\alpha - x_0) = y.$$

We shall construct the sequence x_α in the following way. By virtue of (4.3) and (4.4), for each positive integer α we can select a value of j, which we call j_α, such that both inequalities

$$\|x_{\alpha j_\alpha} - x_0\| < \frac{1}{\alpha}$$

and

$$\|c_{\alpha j_\alpha}(x_{\alpha j_\alpha} - x_0) - y_\alpha\| < \frac{1}{\alpha}$$

hold. Now we put

$$x_\alpha = x_{\alpha j_\alpha}; \quad \gamma_\alpha = c_{\alpha j_\alpha}.$$

Obviously, the first of the inequalities (4.5) holds. For showing that the second holds too, we write

$$\|\gamma_\alpha(x_\alpha - x_0) - y\| \leq \|\gamma_\alpha(x_\alpha - x_0) - y_\alpha\| + \|y_\alpha - y\|$$

and see that for any $\epsilon > 0$, we have $\|y_\alpha - y\| < \frac{\epsilon}{2}$ and $\|\gamma_\alpha(x_\alpha - x_0) - y_\alpha\| < \frac{\epsilon}{2}$ for all α greater than some $N(\epsilon)$.

Lemma 4.3: For any set A (and any point x_o)

$$T[\partial A] \subseteq T[A].$$

The proof is similar to the preceding one. Suppose that $y \in T[\partial A]$. There exist sequences $c_i > 0$ and $x_i \to x_o$ such that $c_i(x_i - x_o) \to y$. Furthermore, as $x_i \in \partial A$, we may write

$$x_i = \lim_{j \to \infty} x_{ij}, \quad x_{ij} \in A.$$

For each positive integer α we can select a value of $j = j_\alpha$ such that

$$\|x_{\alpha j_\alpha} - x_\alpha\| = o(\|x_\alpha - x_o\|).$$

Hence

$$\lim_{\alpha \to \infty} (x_{\alpha j_\alpha} - x_\alpha) = \lim_{\alpha \to \infty} (x_\alpha - x_o) = 0$$

and

$$\lim_{\alpha \to \infty} c_\alpha(x_{\alpha j_\alpha} - x_o) = \lim_{\alpha \to \infty} c_\alpha(x_\alpha - x_o) = y,$$

this last relation being valid because $x_{\alpha j_\alpha} - x_o = (x_{\alpha j_\alpha} - x_\alpha) + (x_\alpha - x_o)$ is an infinitesimal which is equivalent to $x_\alpha - x_o$. So, since in the last equality, $x_{\alpha j_\alpha} \in A$, we conclude that $y \in T[A]$.

Corollary: $T[\bar{A}] = T[A]$, where $\bar{A}$ denotes the closure of A.

Proof:

$$T[\bar{A}] = T[A \cup \partial A] = T[A] \cup T[\partial A] = T[A].$$

Theorem 4.1: For any set A

$$\partial T[A] \subseteq T[\partial A].$$

We start proving an auxiliary lemma.

Lemma 4.4: If Y_o is an open cone (a cone projecting an open set), if A is a given set and if y_o is a non-zero vector fulfilling the following conditions:

$$\text{i)} \quad y_o \in Y_o$$

$$\text{ii)} \quad y_o \in T_{x_o}[A]$$

then, for any $\epsilon > 0$ there exists a point $x \in A$ such that $0 < \|x - x_o\| \leq \epsilon$ and $x - x_o \in Y_o$.

Indeed, since $y_o \in T_{x_o}[A]$, there exists a sequence $x_i \to x_o$, $x_i \in A$, $c_i(x_i - x_o) \to y_o$, where $c_i > 0$. Therefore, $\|x_i - x_o\| \leq \epsilon$ for i greater than some N_1. Since Y_o is open, $c_i(x_i - x_o) \in Y_o$ for i greater than some N_2. Taking $i > \max(N_1, N_2)$, x_i fulfills the desired requirements.

Now we return to the proof of Theorem 4.1. It is easy to see that $\partial T[A]$ is a cone, i.e., that if $y_o \in \partial T[A]$ then the whole ray $\{cy_o | c > 0\}$ belongs to $\partial T[A]$. Let us suppose that

$$(4.6) \qquad\qquad y_o \in \partial T_{x_o}[A];$$

if we show that y_o necessarily belongs to $T_{x_o}[\partial A]$, the theorem will be proved.

We take a sequence of open connected cones Y_i such that $Y_{i+1} \subseteq Y_i$ and $\bigcap_{i=1}^{\infty} Y_i$ is the single ray cy_o, $(x > 0)$. We may take, for example, on the unit sphere the sets $\{y | \|y - \frac{y_o}{\|y_o\|}\| < \varepsilon_i\}$ with $\varepsilon_i > 0$, $\varepsilon_i \to 0$, and project them from the center of the sphere.

Let us call now Y_i^* the intersection of the cone Y_i with the ball of radius ε_i, i.e.,

$$Y_i^* = \{y \mid y \in Y_i; \ \|y\| < \varepsilon_i\},$$

where always $\varepsilon_i \to 0$ (see Figure 2).

Since $y_o \in \partial T_{x_o}[A]$, in any open cone Y_i there exists two rays, one belonging to $T[A]^o$ and the other to its complement $CT[A]$. From the preceding lemma we deduce, therefore, that in any neighborhood of the first ray there is a point $x' \neq x_o$, such that $x_i' \in A$, $x_i' - x_o \in Y_i^*$.

Figure 2.

On the second ray there exists some point $x_i'' \neq x_o$ such that $x_i'' \notin A$ and $x_i'' - x_o \in Y_i^*$ (indeed, a whole segment of this ray, starting at x_o, does not belong to A, otherwise this ray would belong to $T[A]$.) Therefore, the set $Y_i^* \cdot \{x_o\}$ being connected, there exists a point $x_i \neq x_o$ such that $x_i \in \partial A$, $x_i - x_o \in Y_i^*$ and $\|x_i - x_o\| < \varepsilon_i$. From here we conclude that

$$\text{i)} \quad x_i \to x_o \text{ as } i \to \infty;$$

$$\text{ii)} \quad c_i(x_i - x_o) \to y_o, \text{ if we take } c_i = \frac{\|y_o\|}{\|x_i - x_o\|} > 0.$$

Remark. It is important to note that the converse is not necessarily true, that is,

$$T[\partial A] \text{ not necessarily } \subseteq \partial T[A]$$

as shown in an example, in Figure 3.

Figure 3

5. The reachable cone.

<u>Definition</u>: Given equation (1.5) and a certain point x_0, let R_{x_0} be the reachable set from x_0; we shall call "reachable cone from the point x_0 at the point x", and denote by $C_{x_0}[x]$ or simply $C[x]$, the tangent cone of R_{x_0} at the point x:

$$(5.1) \qquad C_{x_0}[x] = T_x[R_{x_0}].$$

Taking into account that R_{x_0} is closed and connected, it is easily seen that a necessary and sufficient condition for $C_{x_0}[x]$ to be not empty is that $x \in R_{x_0}$.

<u>Theorem 5.1</u>: Let $u(t)$ be an admissible control function of equation (1.5), defined in the interval (t_1, t_2) where $t_1 < t_2$, $x(t)$ the corresponding trajectory, let $C[x(t)] = C_{x_0}[x(t)]$ defined as above for some point x_0, and let $A(t, t_1)$ be the matrix defined in (3.7) as solution of the variational equation along $x(t)$; then

$$(5.2) \qquad C[x(t_2)] \supset A(t_2, t_1)C[x(t_1)].$$

<u>Proof</u>: Suppose $y \in C[x(t_1)]$, so that there exists a sequence $x_1 \to x(t_1)$, $x_1 \in R$, $c_1(x_1 - x(t_1)) = y_1 \to y$, $c_1 > 0$. If we show that $A(t_2, t_1) \cdot y \in C[x(t_2)]$, we will have proved the theorem.

Let $x_1^!$ be the point obtained by translation of x_1 along the trajectory corresponding to the same control function $u(t)$ during the time interval (t_1, t_2) (similarly as in the proof of Theorem 3.1). Let $y_1^!$ be the value for $t = t_2$ of the solution of the variational equation starting at y_1 for $t = t_1$, so that (see Figure 4)

$$(5.3) \qquad y_1^! = A(t_2, t_1)y_1.$$

Between $x_1^!$ and $y_1^!$ exists the relation

$$x_1^! - x(t_2) = \frac{y_1^!}{c_1} + o(\|x_1 - x(t_1)\|).$$

Figure 4.

Therefore, supposing the existence of these limits,

$$(5.4) \qquad y^! = \lim y_1^! = \lim c_1(x_1^! - x(t_2)).$$

Since the linear operator $A(t_2, t_1)$ is continuous, we obtain from (5.3)

$$y^! = \lim y_1^! = A(t_2, t_1)\lim y_1 = A(t_2, t_1)y$$

which, by the way, proves the existence of the limit. Moreover, from the way they are obtained, it follows that $x_1^! \in R$, hence (5.4) proves that $A(t_2, t_1)y \in C[x(t_2)]$, which was our task.

<u>Theorem 5.2</u>: If $x(t)$ is the trajectory of (1.5) corresponding to the admissible control function $u(t)$, defined in the interval (t_0, t_1), if $x(t_0) = x_0$, $x(t_1) = x_1$; and if at the point x_1 the tangent cone $C_{x_0}[x_1]$ is contained in the half-space defined by $p_1 \cdot y \geq 0$, where

p_1 is a covariant (row-)vector, then, at any point $x(t)$ with $t_0 \leqq t \leqq t_1$, the cone $C_{x_0}[x(t)]$ is contained in the half-space $p(t) \cdot y \geqq 0$ where $p(t) = p_1 \cdot A(t_1, t)$ is the solution of the adjoint variational equation.

Proof: If for some $y_t \in C[x(t)]$ it would be $p(t) \cdot y_t = p_1 \cdot A(t_1, t)y_t < 0$, applying the preceding theorem we see that

$$y_1 = A(t_1, t)y_t \in C[x_1]$$

and $p_1 y_1 < 0$, contradicting our hypothesis.

This theorem generalizes the theorem (3.2), and here the advantage is that we are not assuming any kind of smoothness of the boundary of R_{x_0}.

Corollary 5.1: If, under the conditions of the last theorem, the reachable set R_{x_0} is "flat" in the neighborhood of the point x_1, this is if $C_{x_0}[x_1]$ is contained in an r-dimensional linear subspace, the same happens along the trajectory $x(t)$ $(t_0 \leqq t \leqq t_1)$.

Corollary 5.2: If, under the same conditions $C_{x_0}[x_1]$ has a convex edge or, more generally, a convex vertex:

$$p_1 \cdot y \geqq 0, \ldots, p_r' y \geqq 0; \quad y \in C_{x_0}[x_1],$$

the same happens along the whole trajectory $x(t)$ $(t_0 \leqq t \leqq t_1)$.

When $C_{x_0}[x]$ is convex for all x, as in the linear case, we obtain the result that any trajectory on the boundary of R_{x_0} cannot move from a smooth to a "rougher" part of ∂R (it can move from a vertex to an edge and from there to a region where ∂R has a tangent plane, but never in the opposite direction).

6. The admissible cone.

Definition: Given equation (1.5) and the set U of admissible values of the variable u, we shall call "admissible cone at point x_0", and denote by $F[x_0]$, the cone which projects the set $f(x_0, U)$; this is

$$y \in F[x_0]$$

it and only if there exists a value $u_0 \in U$ and a positive number c such that

(6.1) $$y = cf(x_0, u_0).$$

$F[x]$ is closed and convex, since the set $f(x, U)$ has those properties.

Note that $f(x, U)$ is the section of the cone $F[x]$ with the plane $y^0 = 1$.

Our task now is to study the relation between the admissible cone $F[x]$ and the reachable set R, and especially its tangent cone $T_x[R]$.

Lemma 6.1: If, under the conditions stated in Section 1, $u(t)$ is an admissible control function and $x(t)$ the corresponding trajectory of (1.5), then

(6.2) $$\frac{\Delta x}{\epsilon} = \frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} f(x(t), u(t))dt$$

is a vector belonging to the convex hull of the set $\{f(x, U)|x = x(t); t_0 \leq t \leq t_0 + \epsilon\}$.

This is a vectorial generalization of the classical mean value theorem. For proving it we multiply (6.2) by any vector p, obtaining

$$p \cdot \frac{\Delta x}{\epsilon} = \frac{1}{\epsilon} \int_{t_o}^{t_o + \epsilon} p \cdot f(x(t), u(t)) dt.$$

Now the integrand is a scalar function, so that we have

(6.3) $\text{l.u.b.} \{p \cdot f(x(t), U)\} \geq p \cdot \frac{\Delta x}{\epsilon} \geq \text{g.l.b.} \{p \cdot f(x(t), U)\}$

for $0 \leq t - t_o \leq \epsilon$. Since these inequalities are valid for any vector p, they define precisely the convex hull of the set $f(x(t), U)$, for $t_o \leq t \leq t_o + \epsilon$.

Lemma 6.2: Let

(6.4) $\delta = \delta(x_o, \epsilon) = \text{l.u.b.} \{\|f(x, u) - f(x_o, u)\|\}$

subject to the conditions

(6.4') $\|x - x_o\| \leq \epsilon; \quad u \in U.$

Then, under the conditions of §1,

(6.5) $\delta(x_o, \epsilon) \to 0 \quad \text{for} \quad \epsilon \to 0.$

Proof: As $\delta(x_o, \epsilon)$ is a nondecreasing function of ϵ for $\epsilon > 0$, there exists the limit

$$l = \lim_{\epsilon \to 0} \delta(x_o, \epsilon).$$

Let us take the sequences of positive numbers $\mathcal{E}_i \to 0$, $\rho_i \to 0$ for $i = 1, 2, 3, \ldots$. For any i we can select a value $u_i \in U$ and a value x_i such that $\|x_o - x_i\| \leq \mathcal{E}_i$ and

$$\delta(x_o, \mathcal{E}_i) \geq \|f(x_i, u_i) - f(x_o, u_i)\| \geq \delta(x_o, \mathcal{E}_i) - \rho_i.$$

As $i \to \infty$, $\rho_i \to 0$ and therefore

$$\|f(x_i, u_i) - f(x_0, u_i)\| \to \mathit{l}.$$

According to (6.4'), $x_i \to x_0$. Since U is compact, we can select a subsequence such that the $\lim u_i = u_0$ exists; we shall suppose that $i = 1,2,3, \ldots,$ refers to that subsequence. Hence, the continuity of $f(x, u)$ for $x = x_0$ assumes that

$$\mathit{l} = \lim_{i \to \infty} \|f(x_i, u_i) - f(x_0, u_i)\| = 0.$$

With these elements we can prove for rather general conditions, what in the simplest cases is quite obvious, namely that "in first approximation" the reachable points from a given x_0, are those which are inside the cone $F[x_0]$.

Theorem 6.1: If the differential equation (1.5) satisfies the conditions of §1, then

$$(6.6) \qquad T_{x_0}[R_{x_0}] = F[x_0].$$

Proof: First we shall prove that if $y \in F[x_0]$, then $y \in T_{x_0}[R_{x_0}]$. This is easy, because in this case there exists a positive number c and a value $u_0 \in U$ such that $c \cdot f(x_0, u_0) = y$, and if we integrate the equation (1.5) with $u(t) = u_0 = $ const. and the initial condition $x(t_0) = x_0$, we obtain a trajectory $x(t)$ belonging evidently to R_{x_0}, for which the vector y is tangent at the point x_0, and therefore $y \in T_{x_0}[R_{x_0}]$.

For proving the converse, we suppose that $y \in T_{x_0}[R_{x_0}]$. Hence, there exist sequences $c_i > 0$, $x_i \in R_{x_0}$ $(i = 1,2,3, \ldots)$ such that

$$x_0 = \lim x; \quad y = \lim c_i(x_i - x_0).$$

The relation $x_1 \in R_{x_0}$ implies the existence of control functions $u_1^*(t)$ and corresponding trajectories of the differential equation $x_1^*(t)$, leading from $x_1^*(t_0) = x_0$ to $x_1^*(t_0 + \mathcal{E}_1) = x_1$, $\mathcal{E}_1 > 0$. We now apply Lemma 6.1 to the equality

$$x_1 - x_0 = \int_{t_0}^{t_0 + \mathcal{E}_1} f(x_1^*(t), u_1^*(t))dt,$$

concluding that $\dfrac{x_1 - x_0}{\mathcal{E}_1}$ belongs to the convex hull of $\{f(x, U) \mid \|x - x_0\| < \eta_1\}$, where $\eta_1 = \eta_1(_1)$ might be taken of the form $\eta_1 = K \cdot \mathcal{E}_1$, taking for K an upper bound of $\|f(x, u)\|$ in a suitable neighborhood of x_0. The fact that the component $f^o(x, u) \equiv 1$ assures that $\left\| \dfrac{x_1 - x_0}{\mathcal{E}_1} \right\| \geq 1$ and therefore $\mathcal{E}_1 \to 0$ for $x_1 \to x_0$.

It is easy to see that the sequence $\dfrac{x_1 - x_0}{\mathcal{E}_1}$ contains a convergent subsequence; in fact, it is contained in the convex hull of $\{f(x, u) \mid \|x - x_0\| < \max \eta_1\}$, which is a bounded set. Let us suppose in the following that the indices $i = 1, 2, 3, \ldots$ refer to that convergent subsequence.

From the relation of Lemma 6.1 we deduce that for any vector p

$$(6.7) \quad \text{g.l.b.}\{p \cdot f(x, U) \mid \|x - x_0\| < \eta_1\} \leq p \cdot \frac{x_1 - x_0}{\mathcal{E}_1} < \text{l.u.b.}\{p \cdot f(x, U) \mid \|x - x_0\| < \eta_1\}$$

this following from the fact that

$$\text{g.l.b.}\{p \cdot \text{convex hull } \{f(x, U) \mid \|x - x_0\| < \eta_1\}\} =$$

$$= \text{g.l.b.}\{p \cdot \{f(x, U) \mid \|x - x_0\| < \eta_1\}\}$$

and similarly for the l.u.b.

We may write

$$\text{l.u.b.}\{p \cdot f(x, U) \mid \|x - x_o\| < \eta_i\} \leq \text{l.u.b.}\{p \cdot f(x_o, U)\} +$$

$$+ \text{l.u.b.}\{p[f(x, u) - f(x_o, u)] \mid u \in U, \|x - x_o\| < \eta_i\}$$

and

$$\text{g.l.b.}\{p \cdot f(x, U) \mid \|x - x_o\| < \eta_i\} \geq \text{g.l.b.}\{p \cdot f(x_o, U)\} -$$

$$- \text{l.u.b.}\{p[f(x, u) - f(x_o, u)] \mid u \in U, \|x - x_o\| < \eta_i\}.$$

By Lemma 6.2, the last term in both inequalities tends to zero for $i \to \infty$, therefore taking limits in (6.7) we get

$$(6.8) \quad \text{g.l.b.}\{p \cdot f(x_o, U)\} \leq p \cdot \lim \frac{x_i - x_o}{t_i} \leq \text{l.u.b.}\{p \cdot f(x_o, U)\}.$$

Since $\{f(x_o, U)\}$ is compact and convex by hypothesis, and (6.8) holds for any vector p,

$$(6.9) \quad \lim \frac{x_i - x_o}{\mathcal{E}_i} \in f(x_o, U).$$

Besides, since $\left\| \dfrac{x_i - x_o}{\mathcal{E}_i} \right\| \geq 1$ and bounded, its limit is neither zero nor infinity, therefore

$$y = \lim c_i(x_i - x_o) = (\lim c_i \mathcal{E}_i)(\lim \frac{x_i - x_o}{\mathcal{E}_i}) = k \lim \frac{x_i - x_o}{\mathcal{E}_i}$$

which, together with (6.9) implies that $y \in F[x_o]$ by definition of the admissible cone.

This theorem can be applied to compare the relative position of the admissible cone $F[x_1]$ at any other point than x_o. Indeed, $x_1 \in R_{x_o}$

implies $R_{x_1} \subseteq R_{x_0}$ and $T_{x_1}[R_{x_1}] \subseteq T_{x_1}[R_{x_1}]$. Applying the last theorem we obtain immediately the following theorem.

Theorem 6.2: If the differential equation (1.5) satisfies the conditions of §1, and the point x_1 is reachable from x_0, then

$$(6.10) \qquad F[x_1] \subseteq T_{x_1}[R_{x_0}].$$

These theorems do not assert anything about the actual reachability of the points located in the admissible cone. This will be the concern of the following ones.

Theorem 6.3: If, under the assumptions of Theorem 6.1, Y is a closed cone belonging to the interior of the cone $F[x_0]$, then there exists an $\eta > 0$ such that all points of the form

$$\{x_0 + y \mid y \in Y, \ \|y\| < \eta\}$$

are reachable from x_0.

Proof: If $x(t)$ is the solution of the differential equation corresponding to the control function $u(t) = u = \text{const.}$, starting at x_0, we may write

$$\left\| \int_{t_0}^{t_0+\varepsilon} [f(x(t), u) - f(x_0, u)] dt \right\| \leq K \int_{t_0}^{t_0+\varepsilon} \|x(t) - x_0\| dt \leq$$

$$\leq K \int_{t_0}^{t_0+\varepsilon} dt \int_{t_0}^{t} \|f(x(t), u)\| dt \leq K M \frac{\varepsilon^2}{2},$$

where K and M are suitable constants which can be taken independently of u for all $u \in U$ (since the set U is compact), provided t is restricted to some neighborhood of zero. Therefore

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} f(x(t),\, u)dt = f(x_0,\, u)$$

uniformly in $u \in U$.

The set $f(x_0,\, U)$ is really n-dimensional (not n+1 dimensional) because its zero component is always 1. We shall consider in what follows the n-dimensional interior of this set, this is the interior according to the topology of the n-dimensional hyperplane $f^0 \equiv 1$ (see fig. 5).

For any fixed ε, the mapping of n-space into n-space, which maps $f(x_0,\, u)$ in $\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} f(x(t),\, u)dt$ is continuous in its argument $f(x_0,\, u)$, therefore it maps the compact set $f(x_0,\, U)$ onto a compact set, and according to what was shown before, this mapping tends uniformly to the identity for $\varepsilon \to 0$. Taking any closed set Y' belonging to the n-dimensional interior of $f(x_0,\, U)$, there exists an $\eta > 0$ such that for every positive $\varepsilon < \eta$ the image of $f(x_0,\, U)$ in the above defined mapping, covers Y'; for seeing this it is sufficient to take η so small that for every pair $(\varepsilon,\, u)$, $0 < \varepsilon < \eta$, $u \in U$ the following inequality holds:

$$\left\| f(x_0,\, u) - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} f(x(t),\, u)dt \right\| < \text{distance from set } Y' \text{ to the n-dimensional boundary of } f(x_0,\, U).$$

Fig. 5.

Now we take as set Y' the intersection of the closed cone Y of our hypothesis with the plane $(x - x_0)^0 = 1$. Since Y is closed and contained in the (n+1 dimensional) interior of $F[x_0]$, also Y' will be closed and contained in the (n-dimensional) interior of $f(x_0, U)$. We may, therefore, apply the preceding result stating that there exists an $\eta > 0$ such that for any $y' \in Y'$ and any positive $\xi < \eta$, there exists an $u \in U$ such that

$$y' = \int_{t_0}^{t_0 + \xi} f(x(t), u)dt.$$

We shall prove that this η fulfills the requirement of the theorem. Indeed, if $y \in Y$ and $\|y\| < \eta$, we take $y' = y/y^0$ and $\xi = y^0 \leq \|y\| < \eta$ (y^0 is the zero component of tne vector y, which is positive because $y \in F[x_0]$). Then $y' \in f(x_0, U)$ and we may apply the last result, obtaining that

$$x_0 + y = x_0 + \xi y' = x_0 + \int_{t_0}^{t_0 + \xi} f(x(t), u)dt$$

for some value of $u \in U$, i.e., $x_0 + y$ is actually reachable from x_0.

Corollary 6.1: If R is the reachable set from the point x_0, Q is the complement of R (i.e., the set of not reachable points) and if x_1 is reachable from x_0, then, under the assumptions of the preceding theorems, the intersection $F[x_1] \cap T_{x_1}[Q]$ is void or belongs to the boundary of $F[x_1]$.

Indeed, if we suppose that y belongs to the interior of $F[x_1]$, there exists a closed cone Y which belongs to the interior of $F[x_1]$ and contains y in its interior. Consider a sequence of points $x_i \to x_1$ ($i = 2,3,4, \ldots$), such that $c_i(x_i - x_1) \to y$, ($c_i > 0$). We shall prove that $x_i \in Q$ is impossible (and there $y \in T_{x_1}[Q]$ is impossible). In fact, $c_i(x_i - x_1) \to y$ and y is interior to the cone Y, therefore $x_i - x_0 \in Y$

for sufficiently large i, and by Theorem 6.3 x_i is reachable from x_1
for i larger than some i_0.

§7. Relations with the boundary of R.

Theorem 7.1: Given the differential equations (1.5) satisfying the conditions stated there, denoting by $R = R_{x_0}$ the reachable set from the point x_0, ∂R its boundary, and x_1 any point, the intersection of the cones $F[x_1]$ and $T_{x_1}[∂R]$ is void or belongs to the boundary of $F[x_1]$.

The proof is immediate if we apply the corollary 6.1, taking into account that by Theorem 5.3,

$$T_{x_1}[∂R] = T_{x_1}[∂Q] \subseteq T_{x_1}[Q]$$

where Q is the complement of R.

Theorem 7.2: If, under the same assumptions, the trajectory $x(t)$ belongs to the boundary of R for $t_0 \le t \le t_2$, and $t_0 \le t_1 < t_2$, then denoting $x(t_1) = x_1$, the cones $F[x_1]$ and $T_{x_1}[∂R]$ have a non-void intersection, which belongs to the boundary of $F[x_1]$.

Taking into account the preceding theorem, it remains only to prove the existence of at least one common ray of $T_{x_1}[∂R]$ and $F[x_1]$. This ray is easily obtained by taking, on the trajectory $x(t)$, a decreasing sequence $t_i \to t_1$ (i = 3,4,5, ...) and the corresponding points $x_i = x(t_i) \in ∂R$. From the sequence

$$c_i(x_i - x_1), \quad c_i = \frac{1}{\|x_i - x_1\|}$$

we take a converging subsequence; its limit belongs to $F[x_1]$ by Lemma 6.1 and a reasoning similar to that of Theorem 6.1. Therefore $y \in F[x_1]$, $y \in T_{x_1}[∂R]$.

From here we get the relation with Pontrjagin's "Maximum Principle". Indeed, this principle applies to optimization problems characterized precisely by the fact that any optimal trajectory (i.e., solution of the optimization problem) lies on the boundary of the reachable set (from the initial point). Any such trajectory has a definite tangent vector at almost every point and by Theorem (7.2) this tangent vector, given by $f(x(t), u(t))$, belongs to the boundary of $F[x(t)]$. The corresponding value of $u(t)$ may or may not belong to the boundary of U, this fact depending on the mapping of U into $f(x, U)$. In any case, since $F[x]$ is convex, there exists a vector $p(t)$ such that $u(t)$ maximizes the scalar product $p(t) \cdot f(x(t), u(t))$ with respect to all possible values of $u \in U$, and this is the maximum principle.

If along the trajectory the boundary of the reachable set R has a tangent plane, then Theorem 3.2 shows that $p(t)$ satisfies the adjoint variational equation (3.10).

If at some point of the trajectory the boundary of R has a tangent plane, Theorem 5.2 shows that the vector $p(t)$ satisfying the maximum principle may be taken to satisfy the adjoint variational equation even in the case when for previous values of the time the trajectory goes along an (r-dimensional) edge of the boundary of R.

The main difference between the approach of Pontrjagin and this paper, is that here we tried to use global properties of the reachable set, while Pontrjagin uses small variations of the optimizing trajectory.

A nice picture of the reachable set R is obtained by interpreting the previous results in terms of wave propagation. We imagine $x_o = (t_o, x_o)$ as the origin of perturbations propagated along all admissible trajectories. Then, for any fixed value of $t > t_o$, the section of R is the perturbed region. The section of the boundary of R is the wave front. The boundary of R, imagined as a surface given by an equation

$$t_{min} = S(x)$$

is an envelope of the cones $F[x]$ and may be interpreted as a (generalized) solution of a partial differential equation of the first order, with $F[x]$ the characteristic cone at each point. This clearly points out the relation with the Hamilton-Jacobi theory.

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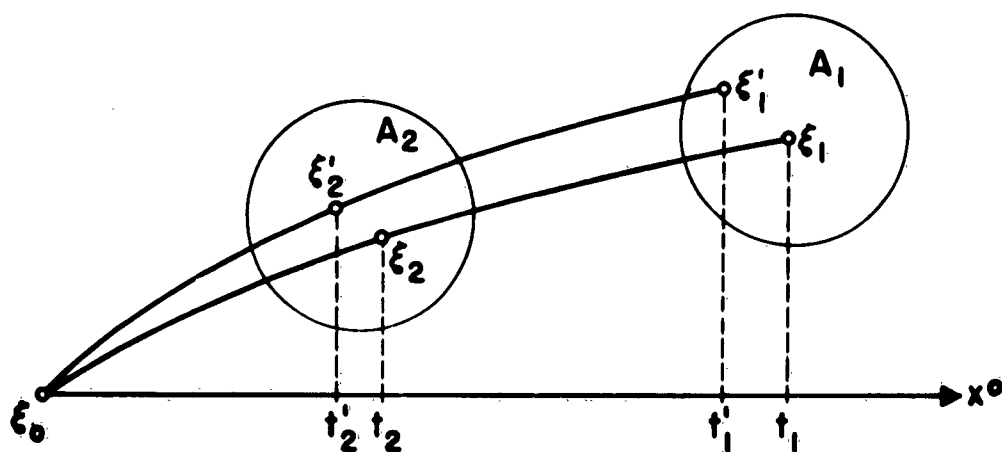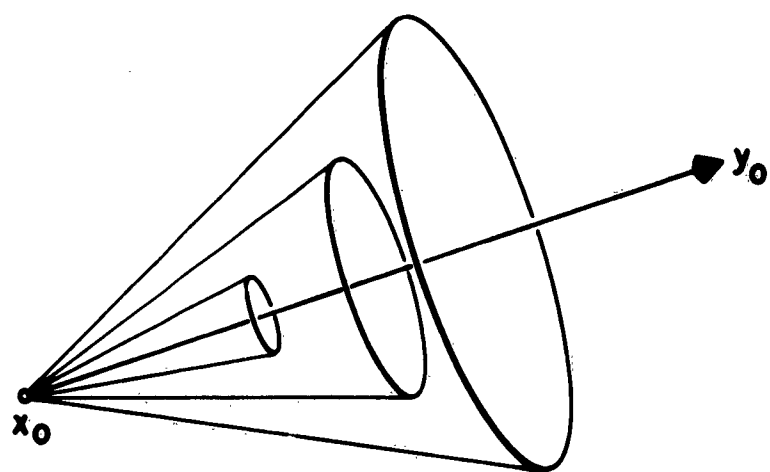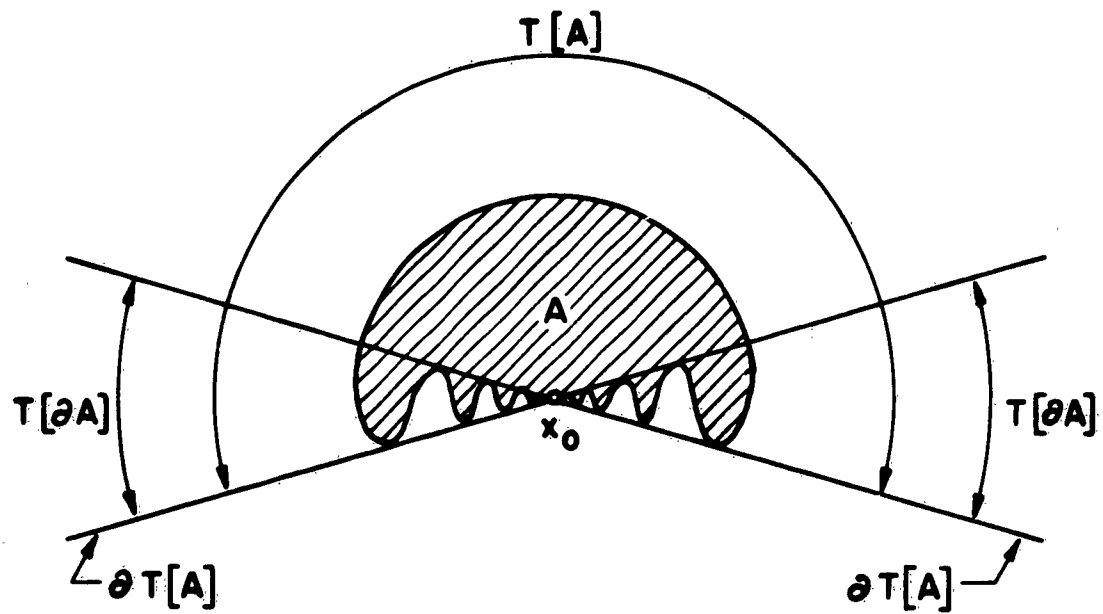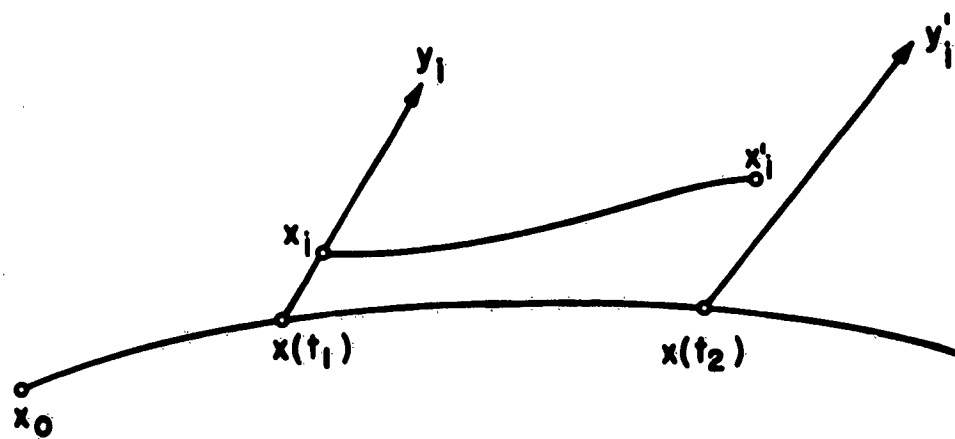   No. 1, 1961, pp. 36-58.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5